EXTENDING THE PERT MODEL FOR PROBABILISTIC ACTIVITY DIRECT COSTS

Mihir Dash, Professor
School of Business
Alliance University, India

Introduction

Project management is the process of designing, planning, and implementing a set of activities to accomplish a particular goal or task (Tache et al., 2013). The two critical performance measures which project managers attempt to control are project duration, i.e. the total time taken to perform all of the activities required to achieve the goal, and project cost, i.e. the total cost incurred in implementing the project. The two most common approaches to project management are the Critical Path Method (CPM; Kelley and Walker, 1959) and the Project Evaluation and Review Technique (PERT; Fazar, 1959).

CPM starts by analysing the project into all the activities/tasks (categorized within a work breakdown structure) required to achieve the goal, each with a definite and known completion time, along with a structure of dependencies/precedence relationships between the activities, and a set of logical endpoints or milestones delineating different stages of the project. This structure of dependencies is modelled as an activity network, either with “activities on arrows” (AOA), in which nodes represent milestones/events, arrows represent activities, and the dependency structure is expressed through incoming and outgoing arrows at each node; or with “activities on nodes” (AON), in which nodes represent activities, and arrows represent dependencies between activities.

CPM first calculates the earliest times at which each activity can start and finish, and the earliest times at which each of the milestones/events would be reached, and identifies the project duration, i.e. the earliest time at which all activities in the project can be completed. CPM also works backwards and calculates the latest times at which each activity should start and finish, and each milestone should be reached, without affecting the project duration. The critical path is identified as the chain of activities joining the starting node of the project with the ending node of the project with longest duration, for which the earliest and latest times coincide.

Activities that are on the critical path are critical activities, which must start and finish at their earliest times, failing which the project will get delayed; the same holds true for milestones on the critical path. On the other hand, activities that are not on the critical path are non-critical, so that they can be delayed to some extent without affecting the project duration.

The total float of an activity is the maximum time that the activity can be delayed without affecting the project duration; however, this leaves no scope for delay in any other activity. The free float of an activity is the maximum time that the activity can be delayed without affecting the project duration and the total float time of all subsequent activities. The independent float of an activity is the maximum time that the activity can be delayed without affecting the project duration and the total float time of all other activities, precedent and subsequent.

The float times of activities play an important role in project scheduling. Critical
activities are scheduled at their earliest times, while activities with independent float can be delayed to the extent of their floats. Resource allocation also plays an important role in project scheduling. There may be some resource constraints to be incorporated in the project schedule, e.g. when the same resource is required by parallel activities. Project managers often try to keep resource requirements as uniform as possible across the project, and this may often affect the project schedule.

CPM also analyses the total project costs. It assumes that activity direct costs are deterministic and inversely linearly related with activity times (within certain ranges), and that the indirect project costs are directly proportional to the project duration. This results in a time-cost trade-off, and CPM identifies which activities can be “crashed”, i.e. reduced in duration at an additional direct cost, in such a way as to minimize the total project cost (comprising total direct costs and total indirect costs). In fact, all of these techniques are (mixed) linear integer/binary programming problems.

PERT starts from a similar set of assumptions as CPM, viz. activities, along with a structure of dependencies/precedence relationships between the activities, and a set of logical endpoints or milestones delineating different stages of the project. However, unlike CPM, PERT assumes probabilistic activity times, based on the beta distribution, which is parametrized by three time estimates: the optimistic time \( t_{opt} \), i.e. the shortest time in which the activity can be completed without any obstacles, the pessimistic time \( t_{pess} \), i.e. the longest time the activity can take in the worst case with maximum obstacles, and the most likely time \( t_{ml} \), i.e. the modal time the activity would take under normal conditions. The expected activity times and variance of the activity times from the beta distribution are then given by the following formulae (Fulkerson, 1962; MacCrimmon and Ryavec, 1962; Martin, 1965):

\[
E(t_{ij}) = \frac{1}{6} \left( t_{opt} + 4t_{ml} + t_{pess} \right),
\]

\[
V(t_{ij}) = \frac{1}{36} \left( t_{pess} - t_{opt} \right)^2.
\]

The expected activity times are subsequently used to identify the critical path. The project duration is assumed to be normally distributed, with the expected project duration given by the sum of the expected activity times on the critical path; further, activity times are assumed to be independent of each other, so that the variance of the project duration given by the sum of the variance of activity times on the critical path.

MacCrimmon and Ryavec (1962) identified some possible sources of errors in the PERT model: from the beta distribution assumption, from the mean and standard deviation formulae, from the imprecise time estimates, and from the network representation of the project. They studied the effect of errors from each of these sources on the mean and standard deviation of the project duration. They found that these assumptions can lead to up to 30% error in the estimation of the mean project duration, and 15% error in the standard deviation of the project duration. They found that the PERT formula/estimate for the mean project duration was always optimistically biased, while the estimate for the standard deviation could be biased in either direction. These errors, both for the mean and standard deviation, tend to be large if there are many non-critical paths with expected duration approximately equal to the expected project duration.

One of the major sources of concern for project managers is project cost uncertainty. Higher cost uncertainty leads to higher probability of cost over-run. Thus,
project cost uncertainty is a very important element in project cost control.

The traditional project management methodologies of CPM and PERT only allow for probabilistic activity times, and consequently probabilistic project duration, which is associated with indirect project costs. In particular, PERT/CPM does not consider probabilistic activity direct costs; activity direct costs are assumed to be definite and known, while activity times are assumed to follow beta distributions, parametrized by the three time estimates (optimistic, pessimistic, and most likely times). As a result, the total project cost is taken to be approximately normally distributed, with mean and variance of the total project cost given by

\[ E(TC) = \sum_{(i,j)} dc_{ij} + IC.E(T) \]
\[ V(TC) = IC^2 V(T) \]

where \( dc_{ij} \) is direct cost of activity \((i,j)\), \( T \) is the project duration, and \( IC \) is the indirect cost per unit of time/project duration.

**Model Development**

The PERT/CPM model of project cost is inadequate in practice. Activity times and activity direct costs are usually probabilistic in nature, and it is often important for project budgeting to understand how sensitive project cost estimates are to variations in activity time and activity direct cost estimates. In this context, the PERT model, which allows an assessment of the sensitivity of project duration to variation in activity times, may be extended to include variations in activity direct costs as well. Indirect costs, on the other hand may arise due to several drivers or bases. For simplicity and convenience, indirect costs are assumed to be deterministic, and independent of activity direct costs. Finally, the inverse relationship between activity direct costs and activity times is replaced by considering independent estimates of activity direct costs and activity times in three scenarios: the optimistic case, the most likely case, and the pessimistic case.

The model proposed is described as follows:

1) the expected activity times and variance of activity times are given by:

\[ E(t_{ij}) = \frac{1}{6} \left( t_{opt} + 4t_{md} + t_{pess} \right) \]
\[ V(t_{ij}) = \frac{1}{36} \left( t_{pess} - t_{opt} \right)^2 \]

where \( t_{opt} \), \( t_{md} \), and \( t_{pess} \) are the optimistic, most likely, and pessimistic time estimates for activity \((i,j)\).

2) the expected project duration is the sum of expected activity times of critical activities:

\[ E(T) = \sum_{(i,j) \text{critical}} E(t_{ij}) \]

3) the variance of the project duration is the sum of variances of activity times of critical activities:

\[ V(T) = \sum_{(i,j) \text{critical}} V(t_{ij}) \]

4) the expected activity direct costs and variance of activity direct costs are given by:
\[ E(dc_{ij}) = \frac{1}{6} (dc_{opt} + 4 dc_{ml} + dc_{pess}) \]
\[ V(dc_{ij}) = \frac{1}{36} (dc_{pess} - dc_{opt})^2 \] (6)

where \((dc_{opt})_{ij}\), \((dc_{ml})_{ij}\), and \((dc_{pess})_{ij}\) are the optimistic, most likely, and pessimistic direct cost estimates for activity \((i,j)\).

5) the total project cost is taken to be approximately normally distributed, with expected total project cost and the variance of the total project cost given by:
\[ E(TC) = \sum_{(i,j)} E(dc_{ij}) + IC.E(T) \]
\[ V(TC) = \sum_{(i,j)} V(dc_{ij}) + IC^2 V(T) \] (7)

where \(IC\) is the indirect cost per unit of project duration.

**A Simple Illustration**

Consider the following simple project shown in the Figure 1, with time estimates and direct cost estimates given in the Table 1.

![Figure 1. Simple Project with Activity Times](image)

<table>
<thead>
<tr>
<th>activity</th>
<th>activity times</th>
<th>activity direct costs</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>(t_{opt})</td>
<td>(t_{ml})</td>
</tr>
<tr>
<td>1 - 2</td>
<td>4</td>
<td>7</td>
</tr>
<tr>
<td>2 - 3</td>
<td>6</td>
<td>12</td>
</tr>
<tr>
<td>2 - 4</td>
<td>3</td>
<td>5</td>
</tr>
<tr>
<td>3 - 4</td>
<td>4</td>
<td>6</td>
</tr>
</tbody>
</table>

The first step is to apply PERT analysis. Using the expected activity times, the critical path is found to be 1 - 2 - 3 - 4, so that the expected value, variance, and standard deviation of the project duration are found as follows: \(E(T) = 25\), \(V(T) = 1^2 + 2^2 + 0.67^2 = 5.44\), and \(\sigma(T) = 2.33\).

The next step is to apply the PERT formulae to obtain the expected value and standard deviation of the activity direct costs. Combining these, the expected value, variance, and standard deviation of the total direct cost are given by: \(E(DC) = 671.6667\), \(V(DC) = 33.33^2 + 29.17^2 + 20.83^2 + 28.33^2 = 3198.61\), and \(\sigma(DC) = 56.56\).

Suppose further that indirect cost \(IC\) is Rs 50 per day. The expected value, variance, and standard deviation of the total project cost are given by: \(E(TC) = E(DC) + IC.E(T) = 1921.67\), \(V(TC) = V(DC) + (IC)^2 .V(T) = 3198.61 + 50^2 \times 5.44 = 16809.72\), and \(\sigma(TC) = 129.65\).

*Volume 6 Number 4 Fall 2017*
Thus, the range for the total project cost is given by: in the best case, $\text{TC}_{\text{min}} = 1080$ ($z = -6.49$), and in the worst case, $\text{TC}_{\text{max}} = 2850$ ($z = 7.16$). The $3\sigma$-limits for the total project cost are given by $E(\text{TC}) \pm 3\sigma(\text{TC}) = [1532.71, 2310.63]$, and the 95% confidence interval for the total project cost is given by $E(\text{TC}) \pm 1.96\sigma(\text{TC}) = [1667.55, 2175.79]$.

The distribution of total project cost can be used to assess the probability of exceeding a given budget. For example, the probability that the total project cost will exceed 2000 is given by:

$$p(\text{TC} > 2000) = p(Z > 0.6048) = 0.272862.$$  

**Discussion**

The proposed model allows analysis of the sensitivity of total project cost with variation in activity direct costs. The proposed model suggests that the traditional PERT/CPM approach underestimates the variability in total project cost. In fact, in the illustration above, 19.03% of the variation in total project cost is due to variation in activity direct costs. More generally, the extent to which the variance of the total project cost is underestimated depends on the extent of the indirect cost and the ratio of the variance of the project duration to the variance of the total direct cost.

The proposed model may be extended further by introducing activity-based costing, whereby indirect costs may be allocated according to each activity. The total project cost would become:

$$\text{TC} = \sum_{(i,j)} \left( dc_{ij} + ic_{ij} t_{ij} \right). \tag{8}$$

The analysis would proceed as above, with a slightly more complicated variance. Other resource requirements and cost drivers can also be treated in a similar procedure. Finally, when actuals become available, these can be incorporated to improve the accuracy of estimates.

A key aspect of the model is the relationship between activity times and activity direct costs. The proposed model assumes independence of activity times and activity direct costs. A different approach is suggested by the CPM cost model, which assumes that activity times and activity direct costs are negatively linearly related. This latter approach would result in a decrease in the variance of the total project cost. In the illustration above, under the assumption of perfectly negatively correlated activity times and activity direct costs, the variance of the total project cost is decreased by 65.7688%. Thus, the degree of correlation between activity times and activity direct costs can have a very large impact on the results and would need to be examined further.

Another essential aspect of the model is the assumption that the total project cost is (approximately) normally distributed. Many studies (e.g. Azarova, 2015; Polat, 2012) have proposed alternative probability distributions for activity times and activity costs, but no conclusive evidence has been found against the normal approximation for total project cost. This also needs to be examined further.

**References**

EXTENDING THE PERT MODEL FOR PROBABILISTIC ACTIVITY DIRECT COSTS

Mihir Dash
Alliance University, India

Abstract
One of the major sources of concern for project managers is project cost uncertainty. Higher cost uncertainty leads to higher probability of cost over-run. Thus, project cost uncertainty is a very important element in project cost control. The traditional project management methodologies of PERT/CPM only allow for probabilistic activity times, and consequently probabilistic project duration, which is associated with indirect project costs. In particular, PERT/CPM does not consider probabilistic activity direct costs; activity direct costs are assumed to be definite and known, while activity times are assumed to follow beta distributions, parametrized by the three time estimates (optimistic, pessimistic, and most likely times). This paper proposes an extension of the PERT/CPM model under the conditions of probabilistic activity direct costs and activity times. Further extensions of the model are also discussed.

Keywords: PERT/CPM model, sensitivity, probabilistic activity direct costs, activity times